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2839 [1920, 274].

By translating the steps of the construction of a regular pentagon from plane geometry into algebra show that one of the fifth roots of unity is equal to

$$\tfrac{1}{4}(\sqrt{5}-1)+\tfrac{1}{4}i\sqrt{10+2\sqrt{5}}.$$

(This problem is proposed for solution in Wilczynski and Slaught, *College Algebra with Applications*, Boston, 1916, p. 193.)

I. Solution by H. L. Olson, University of Michigan.

Consider a circle with center at the origin, O, and unit radius, intersecting the positive x-axis at A and the negative y-axis at D. With center at the mid-point, E, of OD, draw a circular arc passing through A and intersecting the positive y-axis at F. The radius of this circle is $\frac{1}{2}\sqrt{5}$, and hence $\overline{OF} \doteq \frac{1}{2}(\sqrt{5}-1)$. With center A and radius AF, draw a circular arc intersecting the circle with center O at G and H. AG and AH are two sides of a regular pentagon inscribed in the circle O, and $\overline{OG}$ and $\overline{OH}$ are the Argand representations of two of the fifth roots of unity.

Since $AF^2 = 1 + (\frac{1}{2}\sqrt{5} - \frac{1}{2})^2$, the equations of the circles O and A are

$$x^2 + y^2 = 1,$$
$$(x-1)^2 + y^4 = 1 + (\tfrac{1}{2}\sqrt{5} - \tfrac{1}{2})^2.$$

Hence, the coördinates of G and H are

$$x = \tfrac{1}{4}(\sqrt{5}-1), \qquad y = \pm\tfrac{1}{4}\sqrt{10+2\sqrt{5}};$$

and two of the fifths roots of unity are $\frac{1}{4}[(\sqrt{5}-1) \pm i\sqrt{10+2\sqrt{5}}]$.

II. Solution by Otto Dunkel, Washington University.

In texts on geometry a pentagon is usually constructed by dividing the radius OA, here taken as of unit length, at M in extreme and mean ratio. The length OM is laid off twice on the circle as chords giving the points A, K, G; then AK is the side of a regular decagon and AG is the side of a regular pentagon. From the definition of extreme and mean ratio it follows that $OM^2 = OA \cdot MA = 1 - OM$, and hence $OM = \frac{1}{2}(\sqrt{5}-1)$. It easily follows that $MG = OG = 1$ and hence the coördinates of G are $x = \frac{1}{4}(\sqrt{5}-1)$, $y = \sqrt{1-x^2} = \frac{1}{4}\sqrt{10+2\sqrt{5}}$. Hence the complex number represented by G is as stated in the problem.

Also solved by T. M. Blakslee, Arthur Pelletier, A. V. Richardson, C. H. Richardson, and F. L. Wilmer.

2853 [1920, 377]. Proposed by J. S. BROWN, Southwest Texas State Normal College, San Marcos, Texas.

Find the side and apothem of a regular pentagon inscribed in a circle without the use of extreme and mean ratio.

Three Solutions by T. M. Blakslee, Ames, Iowa.

I. Let $x + iy$ be the point on the unit circle whose angle is 36°. Then the side of a regular inscribed pentagon will be $p = 2y$. In the equation $(x+iy)^5 = -1$, the coefficient of i is

$$y(5x^4 - 10x^2y^2 + y^4) = 0.$$

We wish the smaller of the two positive values of y. Therefore we can remove the factor y and if we substitute $1 - y^2$ for x^2, our equation reduces to

$$16y^4 - 20y^2 + 5 = 0.$$

Whence $y^2 = (10 - 2\sqrt{5})/16$, $p = \frac{1}{2}\sqrt{10-2\sqrt{5}}$, and the apothem is $a = \frac{1}{4}(\sqrt{5}+1)$.

II. $p = 2 \sin 36° = 4s\sqrt{1 - s^2}$, where $s = \sin 18° = \cos 72°$. But 72° is four times 18°; therefore s is a root of the equation

$$8s^4 - 8s^2 - s + 1 = 0.$$

The factors $s - 1$ and $2s + 1$ correspond to the angles 90° and $-30°$. Thus we have $4s^2 + 2s - 1 = 0$, and the positive root is $s = \frac{1}{4}(\sqrt{5} - 1)$, from which we get p and a.

III. Let d be the side of the regular decagon inscribed in the same circle; then d being a chord of an arc in a unit circle and p that of twice the same arc,

$$d^2 = 2 - \sqrt{4 - p^2}.$$

Let O be the center of the circle, AB and AC chords equal to p and d, F the foot of the perpendicular from A upon OB, and G the middle point of AB. The four points O, F, G and A are concyclic and $BA \cdot BG = BO \cdot BF$.

Now the triangle AOF is congruent to the right triangle formed by the radius and apothem of the decagon, having the same angles and hypothenuse equal to 1. Therefore $OF = d/2$ and our equation may be written

$$p \cdot \frac{p}{2} = 1\left(1 - \frac{d}{2}\right) \text{ or } p^2 = 2 - d.$$

Eliminating d and removing the factor $p^2 - 3$, we have

$$p^4 - 5p^2 + 5 = 0$$

or, since p is only a very little more than 1, $p^2 = \frac{1}{2}(5 - \sqrt{5})$. This gives p and then a as above.

Also solved by Michael Goldberg, R. M. Marshall, J. Q. McNatt, Arthur Pelletier, A. V. Richardson, and L. S. Shively. Some have interpreted the problem as calling for a geometrical construction rather than an algebraic solution.

2859 [1920, 428]. Proposed by L. S. DEDERICK, U. S. Naval Academy.

Derive an expression for the limit of error in evaluating a definite integral by Simpson's Rule.

I. Solution by A. A. Bennett, University of Texas.

Let the desired definite integral be $\int_{a-h}^{a+h} f(x)dx$, where $h > 0$. If the function $f(x)$ be supposed capable of a Taylor expansion through the term in $(x - a)^3$, with a remainder, a simple solution for the problem is obtained by employing this expansion, the form of the result depending upon the form of the remainder adopted.

The problem as stated should not depend upon the existence of derivatives at any point of the interval. A solution not involving such derivatives is the following:

$$\left| \int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a+h) + 4f(a) + f(a-h)\} \right| \leqq \text{Max}\,\{h|x - a|^2|D(x) - D(a+h)|\}, \quad (1)$$

where $(x - a)^2 D(x) \equiv f(x) - 2f(a) + f(2a - x)$.

If $f(x)$ has a second derivative at $x = a$, this is of course the value approached by $D(x)$ as x approaches a. To prove (1), write

$$g(x) \equiv h(x - a)^2\{D(x) - D(a + h)\}.$$

Expanding and integrating we have

$$\int_{a-h}^{a+h} g(x)dx = 2h\left[\int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a+h) + 4f(a) + f(a-h)\}\right].$$

Since $\left|\int_{a-h}^{a+h} g(x)dx\right| \leqq 2h \text{ Max } |g(x)|$, (1) is proved.

It is to be noted that Max $\{h|x - a|^2|D(x) - D(a + h)|\} = 0$, for polynomials of less than the fourth order, so that the formula given by Simpson's rule is exact in these cases.